



A Simple Cell-Centered Multigrid Method for 3D Interface Problems

J. MOLENAAR

CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

(Received December 1995; accepted January 1996)

Abstract—A multigrid algorithm is presented for cell-centered discretizations of interface problems. Instead of constructing the coarse grid operators by means of the Galerkin approximation, the coarse grid operators are obtained by discretization on the coarse grids. The advantage of this approach is that we obtain *M*-matrices on all grids, and that the sparsity pattern of the fine grid matrix is retained on all grids. Moreover, the coarse grid operators are very easy to construct. Numerical results of several test problems are presented.

Keywords—Multigrid, Finite differences, Discontinuous coefficients, Diffusion equation, Strong anisotropies.

1. INTRODUCTION

We consider the numerical solution of the following second-order elliptic boundary value problem:

$$-\sum_{\alpha=1}^d \frac{\partial}{\partial x_{\alpha}} \left(D_{\alpha} \frac{\partial u}{\partial x_{\alpha}} \right) + Cu = f, \quad \text{in } \Omega \subset \mathbb{R}^d, \quad d = 2, 3, \quad (1)$$

$$a \frac{\partial u}{\partial n} + bu = g, \quad \text{on } \partial\Omega, \quad (2)$$

with Ω the bounded domain, D_{α} the positive diffusion coefficient, C the nonnegative linear source term, n the outward normal vector, $a \geq 0$, $b \geq 0$ and $a + b > 0$. Typically D_{α} and C are discontinuous across internal interfaces, and the diffusion coefficient D contains strong anisotropies. This equation appears in many fields of physics and engineering, for example in petrol reservoir simulation and neutron diffusion problems.

In most of the multigrid methods that have been proposed for this problem (cf. [1-3]) the coarse grid operators are constructed as the Galerkin approximation of the fine grid operator, given the grid transfer operators (automatic prescription). The disadvantage of this approach is that the stencils of the coarse grid operators are often larger than the corresponding fine grid stencil. Especially in three-space dimensions this is problematic: 7-point stencils on the finest grid are turned into 27-point stencils on the coarser grids (see, e.g., [3]). Moreover, it is not guaranteed that the coarse grid matrices have the *M*-matrix property if the fine grid matrix is an *M*-matrix, which makes it difficult to select an appropriate smoothing operator for the coarse grid problems.

These problems are avoided in the cell-centered multigrid method that is presented in this paper. We use coarse grid operators that are based on finite volume discretizations on the coarser grids (cf. [4]). In the constant coefficient case these finite volume discretizations can be formulated as Galerkin-like approximations of the fine grid operator. By using this Galerkin-like

Typeset by $\text{\AA}\text{\AA}\text{\AA}\text{-T}\text{\E}\text{\X}$

construction also in the variable coefficient case we obtain finite volume discretizations on all grids. The coefficients in these coarse grid discretizations are simple arithmetic averages of the corresponding coefficients in the finest grid discretization, which is computationally very efficient. The coarse grid operators thus constructed are M -matrices and all have the same stencil.

This approach turns out to be closely related to earlier work by Khalil and Wesseling [5–7]. With the modification for the case $C > 0$ proposed in [6], the same coarse grid operator is obtained in the interior of the domain. Therefore, this paper can be considered to be an extension of the cell-centered multigrid method presented in [6] to three space dimensions. However, the procedure for defining the coarse grid operator developed here is interesting on its own: it clarifies the modification proposed in [6] and it allows a straightforward extension to systems of equations (see [8]).

2. FINITE VOLUME DISCRETIZATION

For the discretization of (1) we use a standard finite volume discretization. For ease of notation we only state the discretization for $\Omega \subset \mathbb{R}^2$ and assume that Ω can be divided by a regular partitioning in open square cells $\Omega_{i,j}^M$ (with side length h), which resolves the discontinuities in D_α and C . Integration of (1) over a cell $\Omega_{i,j}^M$ yields

$$h \left(F_{i+1/2,j}^M + F_{i,j+1/2}^M - F_{i-1/2,j}^M - F_{i,j-1/2}^M \right) + h^2 C_{i,j}^M u_{i,j}^M = h^2 f_{i,j}^M. \quad (3)$$

The flux $F_{i+1/2,j}^M$ at the cell edge $E_{i+1/2,j}^M$ between $\Omega_{i,j}^M$ and $\Omega_{i+1,j}^M$ is approximated by

$$F_{i+1/2,j}^M = -\frac{2D_{1;i,j}^M D_{1;i+1,j}^M}{D_{1;i,j}^M + D_{1;i+1,j}^M} \frac{u_{i+1,j}^M - u_{i,j}^M}{h} \equiv -D_{i+1/2,j}^M \frac{u_{i+1,j}^M - u_{i,j}^M}{h}, \quad (4)$$

with $D_{1;i,j}^M$ the average of D_1 over $\Omega_{i,j}^M$ ($F_{i,j+1/2}^M$ is treated similarly). The coefficient $D_{i+1/2,j}^M$ at the cell edge $E_{i+1/2,j}^M$ is the harmonic average of the values $D_{1;i,j}^M$ and $D_{1;i+1,j}^M$ (cf. [7]), which are assumed to be positive.

The boundary conditions (2) are discretized by symmetric differencing; e.g., the flux at the boundary edge $E_{i+1/2,j}^M$ is approximated by

$$F_{i+1/2,j}^M = -2D_{1;i,j}^M \frac{g_{i+1/2,j}^M - b_{i+1/2,j}^M u_{i,j}^M}{2a_{i+1/2,j}^M + b_{i+1/2,j}^M h}. \quad (5)$$

After including the boundary conditions in the right-hand side (so in the sequel $g_{i+1/2,j}^M = g_{i,j+1/2}^M = 0$), the discretized problem (3)–(5) can be written as

$$\mathcal{L}^M u^M = \mathcal{D}^M u^M + \mathcal{C}^M u^M = \mathcal{F}^M, \quad (6)$$

where \mathcal{D}^M corresponds to the second-order term in (1) and \mathcal{C}^M to the zeroth-order term. Clearly the linear operator \mathcal{L}^M has a 5-point stencil.

REMARK 1. The matrix of the linear operator \mathcal{L}^M is irreducible, and it is an M -matrix if

$$\max_{E_{i+1/2,j}^M \subset \partial\Omega} a_{i+1/2,j}^M + \max_{E_{i,j+1/2}^M \subset \partial\Omega} a_{i,j+1/2}^M + \max_{\Omega_{i,j}^M \subset \Omega} C_{i,j}^M > 0. \quad (7)$$

3. CELL-CENTERED MULTIGRID

To solve the linear system of equations (6) we consider a cell-centered multigrid method (for an introduction to multigrid methods, see, e.g., [9–11]). The coarse grids are constructed cell-wise: each coarse grid cell is the union of four fine grid cells, which is natural for a finite volume scheme. Thus, we obtain a hierarchy of grids G^1, \dots, G^M , with mesh size $2^{M-\ell}h$ on grid G^ℓ .

Usually the coarse grid operator in linear multigrid algorithms is constructed by first choosing suitable interpolation operators $P_{\ell-1}^\ell$, the prolongation, and $R_\ell^{\ell-1}$, the restriction (which may both be problem dependent), and then defining the coarse grid operator $\mathcal{L}^{\ell-1}$ as the Galerkin approximation of the fine grid operator \mathcal{L}^ℓ :

$$\mathcal{L}^{\ell-1} = R_\ell^{\ell-1} \mathcal{L}^\ell P_{\ell-1}^\ell. \quad (8)$$

The alternative way to construct the coarse grid operators is to discretize the problem also on the coarser grids. This is often done for nonlinear problems where the construction (8) is not feasible; in [12] this approach is applied for interface problems. Disregarding the boundary conditions this implies in the constant coefficient case that the operator on the next to finest grid can be written in the following Galerkin-like way:

$$\mathcal{L}^{M-1} = \frac{1}{2} R_M^{M-1} \mathcal{D}^M P_{M-1}^M + R_M^{M-1} C^M P_{M-1}^M, \quad (9)$$

with P_{M-1}^M the interpolation by piecewise constants, with stencil representation

$$[P] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

and R_M^{M-1} its adjoint. Notice that the part corresponding to the second-order term \mathcal{D} in (9) is multiplied by a factor $1/2$. This is due the fact that the piecewise constant prolongation leads to fluxes on fine grid edges that are twice the fluxes on the corresponding coarse grid edges.

We now construct the coarse grid operator by (9) also in the variable coefficient case. Recursive use of (9) yields

$$\mathcal{L}^\ell = \left(\frac{1}{2}\right)^{M-\ell} R_M^\ell \mathcal{D}^M P_\ell^M + R_M^\ell C^M P_\ell^M, \quad (10)$$

with P_ℓ^M the interpolation by piecewise constants between G^ℓ and G^M , and R_M^ℓ its adjoint.

The coarse grid operator \mathcal{L}^ℓ in (10) can be considered as a finite volume discretization of the form (cf. (3))

$$2^{M-\ell} h \left(F_{i+1/2,j}^\ell + F_{i,j+1/2}^\ell - F_{i-1/2,j}^\ell - F_{i,j-1/2}^\ell \right) + 4^{M-\ell} h^2 C_{i,j}^\ell u_{i,j}^\ell = 4^{M-\ell} h^2 f_{i,j}^\ell. \quad (11)$$

If some coarse grid cell edge $E_{i+1/2,j}^\ell$ consists of $2^{M-\ell}$ finest grid edges $E_{i_k+1/2,j_k}^M$ with coefficients $D_{i_k+1/2,j_k}^M$, $k = 1, \dots, 2^{M-\ell}$ (cf. (4)) then the flux $F_{i+1/2,j}^\ell$ on that coarse grid edge as defined by (10) is

$$F_{i+1/2,j}^\ell = - \left(\frac{1}{2^{M-\ell}} \sum_{k=1}^{2^{M-\ell}} D_{i_k+1/2,j_k}^M \right) \frac{u_{i+1,j}^\ell - u_{i,j}^\ell}{2^{M-\ell} h} \equiv -D_{i+1/2,j}^\ell \frac{u_{i+1,j}^\ell - u_{i,j}^\ell}{2^{M-\ell} h}. \quad (12)$$

So the coarse grid flux $F_{i+1/2,j}^\ell$ is calculated with a diffusion coefficient $D_{i+1/2,j}^\ell$ that is the arithmetic average of the corresponding finest grid coefficients $D_{i_k+1/2,j_k}^M$ ($F_{i,j+1/2}^\ell$ is treated similarly).

The treatment of the boundary conditions requires special care. Suppose that a coarse grid boundary edge $E_{i+1/2,j}^\ell$ consists of $2^{M-\ell}$ boundary edges on the finest grid; then straightforward application of (10) yields

$$F_{i+1/2,j}^\ell = \frac{1}{2^{M-\ell}} \sum_{k=1}^{2^{M-\ell}} \frac{2D_{1;i_k,j_k}^M b_{i_k+1/2,j_k}^M u_{i,j}^\ell}{2a_{i_k+1/2,j_k}^M + b_{i_k+1/2,j_k}^M h}. \quad (13)$$

In numerical experiments it appears that this treatment of the boundary conditions leads to poor multigrid convergence rates in the case of mixed (or Robin) boundary conditions ($ab > 0$). Therefore, we consider a modification of (13). Suppose that a coarse grid boundary edge consists of $2^{M-\ell}$ finest grid edges; then we define the flux on that coarse grid edge by

$$F_{i+1/2,j}^\ell = \sum_{k=1}^{2^{M-\ell}} \frac{2D_{1;i_k,j_k}^M b_{i_k+1/2,j_k}^M u_{i,j}^\ell}{2a_{i_k+1/2,j_k}^M + b_{i_k+1/2,j_k}^M (2^{M-\ell}h)}. \quad (14)$$

The fluxes $F_{i+1/2,j}^\ell$ defined by (13) and (14) are identical if $a_{i_k+1/2,j_k}^M = 0$ or $b_{i_k+1/2,j_k}^M = 0$.

Notice that in the calculation of the flux at a coarse grid edge we only use the values for $D_{\alpha;i,j}^M$ in the cells $\Omega_{i,j}^M$ of the finest grid that are adjacent to that coarse grid edge, so this approach cannot be considered as a homogenization procedure (cf. [12]).

The elements of \mathcal{C}^ℓ , $\ell < M$ are the arithmetic average of elements \mathcal{C}^M . If a coarse grid cell $\Omega_{i,j}^\ell$ on the grid G^ℓ consists of $4^{M-\ell}$ finest grid cells Ω_{i_k,j_k}^M then it follows from (10) that

$$C_{i,j}^\ell = \frac{1}{4^{M-\ell}} \sum_{k=1}^{4^{M-\ell}} C_{i_k,j_k}^M. \quad (15)$$

Here it is interesting to note that this coarse grid operator is nearly identical to the coarse grid operator defined by Khalil and Wesseling (see [5-7]). For the special case $C = 0$ the definition of the coarse grid operator \mathcal{L}^{M-1} given in (9) turns out to be equivalent in the interior of the domain to one they use. In [6] the definition of the coarse grid operator is modified in order to deal with the case $C \geq 0$. This modification boils down to a splitting of the operator in second and lower order terms as in (6), and using a different coarse grid approximation for these two parts. We obtain the same coarse grid operator in the interior of the domain. However, the treatment of the boundary conditions is different.

As the coarse grid operators defined by (11),(12),(14),(15) are equivalent to a finite volume discretization the following result is immediate.

THEOREM 1. *All coarse grid operators \mathcal{L}^ℓ have 5-point stencils, and all coarse grid matrices are M -matrices, provided that the matrix of \mathcal{L}^M is an M -matrix.*

PROOF. Follows from Remark 1 and (11),(12),(14),(15).

As $P_{\ell-1}^\ell$ and $R_\ell^{\ell-1}$ only interpolate constant functions exactly, we have $m_P = m_R = 1$, with m_P and m_R the order of the prolongation and the restriction (cf. [13]), respectively. For second-order differential equations we need $m_P + m_R > 2$; therefore we use a more accurate prolongation operator \tilde{P} in the multigrid algorithm. In the 2D case we consider a prolongation that is based on bilinear interpolation; its stencil representation is

$$[\tilde{P}] = \frac{1}{16} \begin{bmatrix} 1 & 3 & 3 & 1 \\ 3 & 9 & 9 & 3 \\ 3 & 9 & 9 & 3 \\ 1 & 3 & 3 & 1 \end{bmatrix}.$$

In the 3D case we take a prolongation that is based on linear interpolation between a coarse grid cell and its three relevant nearest neighbors. The stencil representation for this prolongation is

$$\begin{aligned} [\tilde{P}]^{+3/2} &= [\tilde{P}]^{-3/2} = \frac{1}{4} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ [\tilde{P}]^{+1/2} &= [\tilde{P}]^{-1/2} = \frac{1}{4} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \end{aligned}$$

with $[\tilde{P}]^\beta$ the representation of the stencil in the z -plane $k + \beta$. In both 2D and 3D we have $m_{\tilde{P}} = 2$, so $m_{\tilde{P}} + m_R = 3$, which suffices.

Next we consider the choice of the smoothing operator. In 2D we use the Incomplete Line Lower Upper (ILLU) decomposition, which is known to be a robust and efficient smoother for problems with strongly anisotropic and discontinuous coefficients (cf. [14]). In the 3D case, alternating plane relaxation appears to be the only robust smoothers for problems with strong anisotropies (cf. [15]); therefore we use alternating plane Gauss-Seidel (APGS) relaxation for three-dimensional problems. The 2D problems in a plane are then solved approximately by the 2D multigrid method.

To estimate the convergence rates of the 3D multigrid algorithm we perform a Fourier two-grid analysis (cf. [13]) for the anisotropic diffusion equation,

$$-\sum_{\alpha=1}^d \frac{\partial}{\partial x_\alpha} \left(D_\alpha \frac{\partial u}{\partial x_\alpha} \right) = f, \quad (16)$$

with $D_\alpha(x_1, x_2, x_3) = D_\alpha$. In Table 1, we show the smoothing factor ρ_{APGS} for APGS and the two-grid convergence factor λ for the 3D multigrid algorithm with a single APGS relaxation sweep. Here it is assumed that the 2D problems in the plane relaxation are solved exactly. From the results in Table 1 we conclude that the coarse grid operator efficiently eliminates the low frequency errors in the case of strong anisotropies.

Table 1. Smoothing factor ρ_{APGS} and two-level convergence factor λ for the anisotropic diffusion model problem.

D_1	D_2	D_3	ρ_{APGS}	λ
10^0	10^0	10^3	0.041	0.041
10^0	10^0	10^2	0.123	0.123
10^0	10^0	10^1	0.122	0.122
10^0	10^0	10^0	0.049	0.058
10^0	10^0	10^{-1}	0.302	0.302
10^0	10^0	10^{-2}	0.356	0.356
10^0	10^0	10^{-3}	0.143	0.143
10^0	10^1	10^{-1}	0.322	0.322
10^0	10^2	10^{-2}	0.030	0.030
10^0	10^3	10^{-3}	0.000	0.000

Finally, a word about the storage requirements for the 3D multigrid algorithm. Suppose that a 3D grid contains N points. Storage of the solution, right-hand side and fine grid operator then requires approximately $6N$ memory units. To perform a plane relaxation we store the ILLU

factorization for all planes $((4/3) * 3N = 4N)$, and the solution $((1/3)N)$, the right-hand sides $((1/3)N)$ and the operator $((4/3)N)$ on the 2D coarse grids. To apply an alternating plane relaxation sweep, we therefore need $3 * (4N + (4/3)N) + 6N + (2/3)N = 22\frac{2}{3}N$ memory units. Altogether the 3D multigrid algorithm requires approximately $(8/7) * 22\frac{2}{3}N = 25\frac{19}{21}N$ memory units.

4. NUMERICAL EXAMPLES

In this section, we present results for some standard test problem from the literature. The convergence rate ρ of a multigrid iteration is estimated by

$$\rho = \left(\frac{\|r^{(10)}\|_{\infty}}{\|r^{(0)}\|_{\infty}} \right)^{1/10},$$

with $r^{(m)}$ the residual after m multigrid cycles.

The test problems P1 and P2 described below are defined on the d -dimensional unit cell, $\Omega = (0, 1)^d$, $d = 2, 3$. The first test problem P1 is the anisotropic diffusion problem (16) with $f = 1$ and mixed boundary conditions,

$$\begin{aligned} \frac{\partial u}{\partial n} &= 0, & \text{for } \min(x_{\alpha}) = 0, \\ \frac{\partial u}{\partial n} + \frac{1}{2D_{\alpha}}u &= 0, & \text{otherwise.} \end{aligned}$$

The second test problem P2 is a generalization of Example 6 from [3]

$$-\sum_{\alpha=1}^d \frac{\partial}{\partial x_{\alpha}} \left(D_{\alpha} \frac{\partial u}{\partial x_{\alpha}} \right) + 10^{-4}u = 0,$$

with homogeneous Neumann boundary conditions ($b = g = 0$) and a point source of unit strength at the origin. Different (constant) values are given for D_{α} in different subdomains $\Omega_i \subset \Omega$: $D_{\alpha}(x) = D_{\alpha;i}$ for $x \in \Omega_i$.

4.1. Two-Dimensional Results

As our algorithm is closely related to the one proposed in [6], we repeated the numerical experiments presented in that paper. Using the same smoother, symmetric point Gauss-Seidel, we obtained comparable results. Therefore, it can be concluded that the different treatment of the boundary conditions has only a small influence. A harder test problem is posed by the so-called inhomogeneous staircase problem (see [1, Problem IV]). For this problem the algorithm fails to converge with the Gauss-Seidel smoother. Therefore, we switch to ILLU smoothing. In Table 2 the convergence rates are shown for both V -cycles and W -cycles, and different smoothing strategies: ν_1 and ν_2 denote the different number of pre- and post-smoothing sweeps, respectively. With this more powerful smoothing operator the multigrid algorithm is robust indeed. In the sequel we only use ILLU as the smoothing operator for 2D problems.

Table 2. Multigrid convergence factor ρ for inhomogeneous staircase problem.

ν_1, ν_2	V -cycles	W -cycles
0,1	0.22	0.23
1,0	0.34	0.30
1,1	0.15	0.16

Table 3. Multigrid convergence factor ρ for the 2D anisotropic diffusion equation on different grids.

		V-cycles			W-cycles		
D_1	D_2	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
10^0	10^3	0.11	0.13	0.17	0.11	0.13	0.14
10^0	10^2	0.14	0.15	0.17	0.12	0.12	0.12
10^0	10^1	0.12	0.12	0.13	0.09	0.09	0.09
10^0	10^0	0.08	0.10	0.17	0.07	0.07	0.07

For the anisotropic diffusion problem P1 we use the multigrid algorithm with $\nu_1 = 1$ and $\nu_2 = 0$. The convergence rates for different mesh sizes and multigrid strategies are shown in Table 3. We find a good grid independent convergence rate for all test cases.

The second test problem P2 is defined by

$$\Omega_1(\omega) = \{(x, y) \mid 0 < x < \omega, 0 < y < \omega\},$$

$$\Omega_2(\omega) = \{(x, y) \mid \omega < x < 1, 0 < y < \omega\},$$

$$\Omega_3(\omega) = \{(x, y) \mid 0 < x < \omega, \omega < y < 1\},$$

$$\Omega_4(\omega) = \{(x, y) \mid \omega < x < 1, \omega < y < 1\},$$

and

$$D_{1;i} = \begin{cases} 1, & \text{for } i = 1, 3, \\ 0.01, & \text{otherwise,} \end{cases}$$

$$D_{2;i} = \begin{cases} 100, & \text{for } i = 1, 2, \\ 0.01, & \text{otherwise.} \end{cases}$$

We consider different positionings ω for the interface. For this problem it appears necessary to add a single ILLU-sweep for postsmoothing (so $\nu_1 = 1, \nu_2 = 1$); otherwise the multigrid iteration diverges in some cases. In Table 4 we show the worst convergence rates for different mesh sizes and the positioning of the interface ω in that case. For the V-cycle the convergence seems to be h -dependent, although the convergence rate is still acceptable. With the W-cycle the algorithm converges fast and in a grid independent way, even though we are not approximating the solution of a single partial differential equation.

Table 4. Multigrid convergence factor ρ for problem P2 in two dimensions.

		V-cycles		W-cycles	
h	ω	ρ	ω	ρ	
$\frac{1}{16}$	$\frac{5}{16}$	0.15	$\frac{4}{16}$	0.07	
$\frac{1}{32}$	$\frac{9}{32}$	0.26	$\frac{8}{32}$	0.08	
$\frac{1}{64}$	$\frac{18}{64}$	0.43	$\frac{12}{64}$	0.09	

4.2. Three-Dimensional Results

In the 3D multigrid algorithm we use the APGS smoother with a red-black ordering of the planes. The 2D problems that appear in the plane relaxation are solved approximately by means of a single V-cycle of the 2D multigrid algorithm. In this V-cycle we use in all cases one ILLU-sweep for prerelaxation and no postrelaxation; reversing this order leads in some cases to poor

convergence. The application of more multigrid cycles or relaxation sweeps does not improve the convergence substantially in any of the test cases we have tried. This conforms with results known from Fourier analysis (see [16]).

For the anisotropic diffusion model problem P1 we use a single APGS sweep for postrelaxation and no prerelaxation. The convergence rates for this problem are shown in Table 5. We find good, grid independent convergence rates for all test cases, which are in qualitative agreement with the results from the Fourier two-grid analysis (see Table 1).

Table 5. Multigrid convergence factor ρ for the 3D anisotropic diffusion equation.

D_1	D_2	D_3	V-cycles			W-cycles		
			$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
10^0	10^0	10^3	0.13	0.14	0.14	0.10	0.09	0.09
10^0	10^0	10^2	0.13	0.14	0.14	0.09	0.09	0.08
10^0	10^0	10^1	0.10	0.12	0.12	0.08	0.08	0.07
10^0	10^0	10^0	0.09	0.10	0.10	0.07	0.08	0.07
10^0	10^0	10^{-1}	0.16	0.18	0.20	0.15	0.14	0.14
10^0	10^0	10^{-2}	0.17	0.22	0.21	0.15	0.18	0.19
10^0	10^0	10^{-3}	0.11	0.14	0.23	0.11	0.17	0.19
10^0	10^1	10^{-1}	0.19	0.21	0.22	0.16	0.15	0.15
10^0	10^2	10^{-2}	0.18	0.23	0.22	0.16	0.18	0.19
10^0	10^3	10^{-3}	0.09	0.13	0.21	0.09	0.15	0.17

The last test problem is problem P2 in three dimensions (cf. [3]):

$$\begin{aligned}
\Omega_1(\omega) &= \{(x, y, z) \mid 0 < x < \omega, 0 < y < \omega, 0 < z < \omega\}, \\
\Omega_2(\omega) &= \{(x, y, z) \mid \omega < x < 1, 0 < y < \omega, 0 < z < \omega\}, \\
\Omega_3(\omega) &= \{(x, y, z) \mid 0 < x < \omega, \omega < y < 1, 0 < z < \omega\}, \\
\Omega_4(\omega) &= \{(x, y, z) \mid \omega < x < 1, \omega < y < 1, 0 < z < \omega\}, \\
\Omega_5(\omega) &= \{(x, y, z) \mid 0 < x < \omega, 0 < y < \omega, \omega < z < 1\}, \\
\Omega_6(\omega) &= \{(x, y, z) \mid \omega < x < 1, 0 < y < \omega, \omega < z < 1\}, \\
\Omega_7(\omega) &= \{(x, y, z) \mid 0 < x < \omega, \omega < y < 1, \omega < z < 1\}, \\
\Omega_8(\omega) &= \{(x, y, z) \mid \omega < x < 1, \omega < y < 1, \omega < z < 1\},
\end{aligned}$$

and

$$\begin{aligned}
D_{1;i} &= \begin{cases} 1., & \text{for } i = 1, 3, 5, 7, \\ 0.01, & \text{for } i = 2, 4, 6, 8, \end{cases} \\
D_{2;i} &= \begin{cases} 1., & \text{for } i = 3, 4, 7, 8, \\ 100., & \text{for } i = 1, 2, 5, 6, \end{cases} \\
D_{3;i} &= \begin{cases} 0.01, & \text{for } i = 1, 2, 3, 4, \\ 100., & \text{for } i = 5, 6, 7, 8. \end{cases}
\end{aligned}$$

As in the 2D case we have to add a single APGS sweep for presmoothing ($\nu_1 = 1, \nu_2 = 1$) in order to obtain a convergent algorithm for all possible positions ω of the interface. In Table 6 we show the worst convergence rates on different grids, depending on the value of ω . As in the 2D case, the algorithm is not robust for V-cycles, but is for W-cycles: the convergence is fast and appears to be grid independent. It should be noted that the use of W-cycles is not really a drawback in 3D as it is in 2D, because the coarse grids contain relatively fewer points.

Table 6. Multigrid convergence factor ρ for problem P2 in three dimensions.

h	V-cycles		W-cycles	
	ω	ρ	ω	ρ
$\frac{1}{16}$	$\frac{11}{16}$	0.44	$\frac{12}{16}$	0.18
$\frac{1}{32}$	$\frac{15}{32}$	0.60	$\frac{6}{32}$	0.17
$\frac{1}{64}$	$\frac{31}{64}$	0.81	$\frac{56}{64}$	0.17

5. CONCLUSIONS

We have developed a multigrid algorithm for cell-centered discretizations of 3D interface problems. By using a simple construction for the coarse grid operator in the algorithm, we obtain coarse grid matrices that are M -matrices and that have 5- and 7-point stencils in two and three space dimensions, respectively. Away from the boundaries this coarse grid operator is identical to the one defined for the 2D case in [6]. Therefore, our algorithm can be considered to extend this approach in three space dimensions. An alternating plane Gauss-Seidel smoother is used for the 3D algorithm. The problems in the plane are solved approximately by means of one V -cycle with a single ILLU smoothing step. The numerical results for some hard test problems show that the 3D algorithm is robust for W -cycles.

REFERENCES

1. R.E. Alcouffe, A. Brandt, J.E. Dendy, Jr. and J.W. Painter, The multi-grid method for the diffusion equation with strongly discontinuous coefficients, *SIAM J. Sci. Statist. Comput.* **2**, 430–454 (1981).
2. P.M. de Zeeuw, Matrix-dependent prolongations and restrictions in a blackbox multigrid solver, *J. Comput. Appl. Math.* **33**, 1–27 (1990).
3. J.E. Dendy, Jr., Two multigrid methods for three-dimensional problems with discontinuous and anisotropic coefficients, *SIAM J. Sci. Statist. Comput.* **8**, 673–685 (1987).
4. J. Molenaar, A simple and efficient multigrid method for interface problems, In *Computational Methods in Water Resources X*, (Edited by A. Peters), pp. 1425–1431, Kluwer Academic, (1994).
5. M. Khalil, Analysis of linear multigrid methods for elliptic differential equations with discontinuous and anisotropic coefficients, Ph.D. Thesis, TU Delft, (1989).
6. M. Khalil and P. Wesseling, Vertex-centered and cell-centered multigrid for interface problems, *J. Comput. Phys.* **98**, 1–10 (1992).
7. P. Wesseling, Cell-centered multigrid for interface problems, *J. Comput. Phys.* **79**, 85–91 (1988).
8. J. Molenaar, Multigrid methods for fully implicit oil reservoir simulation, Technical Report 95-40, T.U. Delft, (1995); To appear in *Proceedings Copper Mountain Conference on Multigrid Methods 1995*.
9. A. Brandt, Guide to multigrid development, In *Multigrid Methods, Lecture Notes in Mathematics*, (Edited by W. Hackbusch and U. Trottenberg), pp. 220–312, Springer-Verlag, Berlin, (1982).
10. S.F. McCormick, Editor, *Multigrid Methods*, SIAM, Philadelphia, (1987).
11. P. Wesseling, *An Introduction to Multigrid Methods*, John Wiley and Sons, Chichester, (1992).
12. M.G. Edwards and C.F. Rogers, Multigrid and renormalization for reservoir simulation, Presented at the *European Multigrid Conference 1993*, Amsterdam, July 1993.
13. W. Hackbusch, *Multi-Grid Methods and Applications*, Springer Series in Computational Mathematics, Volume 4, Springer-Verlag, Berlin, (1985).
14. R. Kettler, Analysis and comparison of relaxation schemes in robust multigrid and preconditioned conjugate gradient methods, In *Multigrid Methods, Springer Lecture Notes in Mathematics*, Volume 960, (Edited by W. Hackbusch and U. Trottenberg), pp. 502–534, Springer-Verlag, (1982).
15. R. Kettler and P. Wesseling, Aspects of multigrid methods for problems in three dimensions, *Appl. Math. Comp.* **19**, 159–168 (1986).
16. C.A. Thole and U. Trottenberg, Basic smoothing procedures for the multigrid treatment of elliptic 3D operators, *Appl. Math. Comp.* **19**, 333–345 (1986).